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Quantum-mechanical tunneling in associative neural networks

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Abstract. We investigate the quantum-mechanical tunneling between the "patterns" of the, so-called, associative neural networks. Being the relatively stable minima of the "configuration-energy" space of the networks, the "patterns" represent the macroscopically distinguishable states of the neural nets. Therefore, the tunneling represents a macroscopic quantum effect, but with some special characteristics. Particularly, we investigate the tunneling between the minima of approximately equal depth, thus requiring no energy exchange. If there are at least a few such minima, the tunneling represents a sort of the "random walk" process, which implies the quantum fluctuations in the system, and therefore "malfunctioning" in the information processing of the nets. Due to the finite number of the minima, the "random walk" reduces to a dynamics modeled by the, so-called, Pauli master equation. With some plausible assumptions, the set(s) of the Pauli master equations can be analytically solved. This way comes the main result of this paper: the quantum fluctuations due to the quantum-mechanical tunneling can be "minimized" if the "pattern"-formation is such that there are mutually "distant" groups of the "patterns", thus providing the "zone" structure of the "pattern" formation. This qualitative result can be considered as a basis of the efficient deterministic functioning of the associative neural nets.

PACS. 02.50.Ey Stochastic processes – 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion

1 Introduction

Neural networks are currently the most successful model in cognitive neurosciences [1,2]. Presently, the *"physics of neural nets"* distinguishes the two main approaches to the issue. First, and there is a vast literature [1–4] (and references therein) in this concern, there is the "classical" approach, which considers the neural nets as the classic-physics objects, *i.e.*, as the *fully deterministic* physical systems. This approach particularly assumes that the information processing (the dynamics) of the neural nets bears the classic-physics reality, determinism and locality [5].

On the other hand, it is argued [6] that the dynamics of the neural nets should be modeled quantum-mechanically, assuming the *full quantum-mechanical treatment* of the existing "classical" models of the neural nets.

In this paper we argue for another, still not fully explored approach to the issue. Particularly, we propose to consider the neural networks as the *classic-physics systems*, and to *investigate the quantum-mechanical corrections* to their deterministic dynamics, coming from the well-known effect of *quantum-mechanical tunneling*. This is really an "intermediate" approach in physical modeling of the dynamics of neural networks which is not a new idea. Rather, our proposal is analogous to the programs [7] of investigating the *"border territory between quantum and classical"*, still with some special characteristics.

We are particularly concerned with the quantum corrections of the deterministic dynamics (information processing) of the, so-called, *associative (or attractor) neural nets* (*cf.*, *e.g.*, Kohonen [3], Amit [3], Beale and Jackson [4], just to mention some). Our strategy is as follows: we start from the classic-physics models of the associative neural nets, and investigate the process of quantum-mechanical tunneling in the system, posing the question of *"minimizing" the quantum fluctuations*, which are due to the tunneling.

This strategy can be justified as follows.

The *physical states* of the associative neural nets can be represented in the "configuration-energy $(q-V)$" space (*cf.*, *e.g.*, Hopfield [3]) – Figure 1. Each point on the horizontal axis represents a unique configuration, denoted by the vector $\mathbf{q}$. The heights in vertical direction represent the values of the potential energy of each configuration. The vector $\mathbf{q} = (q_1, q_2, \ldots)$ determines the state of the network as a *whole*. Each "coordinate" q_i describes exactly the state of each constituent neuron, or of each the synapse in the network. Further, we shall not specify

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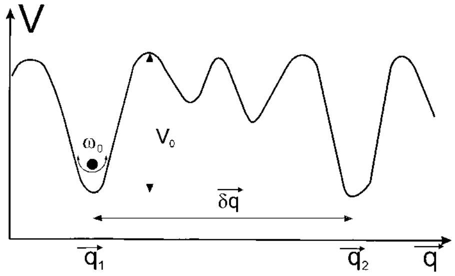

Fig. 1. The local minima of the presented "configuration-energy $(q-V)$" surface represents the "patterns" – well-defined, *macroscopically distinguishable* physical states of the nets. The black ball represents the "particle" oscillating with frequency $\omega_\circ$ around the bottom of the well. Unfortunately, the parameters of the $q-V$ plot, $\omega_\circ$, $V_\circ$, δq, are not generally known.

the meaning of the index "i", assuming applicability of our results to the both, neuronal (short-term) and synaptic (long-term) storage of the informations. For simplicity, we shall further identify $\mathbf{q}$ with its "length", q, thus formally dealing with one-dimensional system.

According to the above assumption, the position of the black ball (the "particle") in Figure 1 should be concerned as a *classical state* of the neural network. This state is unique, in the sense that in an instant t, the state q is known with certainty. This is a basis of *deterministic functioning* of the networks: *the "particle's" trajectory is unique*, bearing unique prescription between the positions, q's, and the instants, t's. Certainly, moving of the ball is driven by external stimuli, and results in the complex process of the "pattern"-formation and/or of the "pattern"-recognition.

More precisely: the dynamics can be considered as a continuous change of the energy surface (plot) by the moving ball, whose movement is driven by external stimuli. As a net effect, the ball finally centers around the bottom of a potential well (*e.g.*, q_1 in Fig. 1), which is a "pattern" – well-defined macroscopic state of the neural network. For some time interval, $\tau_{\rm stab}$, the energy surface ($q-V$ plot) stays intact (unchanged). After this time interval, the "particle" becomes an object of other external stimuli, which force the "particle" to move, and (usually) to change the actual $q-V$ surface. For this, subsequent dynamics, the state q_1 is the initial state.

In general, the $q-V$ plot (surface) has several potential wells (the "patterns"), and at this point arises the *next question*: if the neural networks can exhibit some quantum-mechanical behavior at all, whether the "particle" can "tunnel" between the "patterns", during the time interval $\tau_{\rm stab}$? This question bears some subtleties to be discussed below. Let us assume that the tunneling is possible. Then, according to the (hypothesis of the) *universal validity of quantum mechanics* [5–7,9,11–13], the tunneling appears *unavoidable*. And its consequences for the processing of informations in neural nets are rather obvious: after the tunneling from, *e.g.*, q_1 to q_2, the subsequent dynamics would start from the state q_2, instead of from the – classically unique – state q_1, *thus breaking the classical determinism*. This is really a quantum-mechanical effect, which does not have a classic-physics counterpart, and which causes "malfunctioning" of the neural networks.

At this point arises the question of distinguishing the tunneling from the concurrent the classical effects [7,10]. Namely, the "particle" (*cf.* Fig. 1) can (probabilistically) pick up an amount of energy from the heat bath which could be sufficient for the "particle" to skip out of the potential well, thus, *prima facie*, making the transition (see also Sect. 2) similar to the above distinguished effect caused by the tunneling. With this regard we refer to the Leggett's approach [7], in which it is strongly (and virtually independent on the particular context) stressed that, so as to be able to observe the "pure" tunneling, it is necessary to *suppress the classical activation.* (Typically, it requires very low temperature, and choosing the parameters of the system so as to avoid the applicability of the "correspondence principle".) With this regard, the quantitative results concerning the tunneling in the neural nets *requires* the precise values of the parameters $\omega_\circ, V_\circ, \boldsymbol{\delta}\mathbf{q}$ (*cf.* Fig. 1). Unfortunately this is not generally the case. So, our considerations will be restricted to the general, and rather qualitative considerations, with an emphasis on the possibility to extending the considerations, once the more quantitative results concerning the $q-V$ plot are known.

The plan of this paper is as follows. In Section 2 we show that the problem of the tunneling reduces onto the stochastic process which can be dynamically modeled by the, so-called, Pauli master equation [8]. Due to the fact that $q-V$ plot is rather qualitative, in Section 3 we give analytical solutions for the typical $q-V$ plots. In Section 4 we discuss the results obtained, paying a special attention to the assumptions (and simplifications) underlying the models considered. Section 5 is conclusion.

2 Dynamical model of the tunneling in associative neural networks

It is worth repeating: a "pattern" is a local minimum (potential well) of the $q-V$ configuration (plot) which bears stability for the period of time, $\tau_{\rm stab}$. It is an explicit macroscopic state of a neural network (as a whole) which bears the classical reality and determinism, which particularly guarantees that the "particle" remains captured in the potential well, eventually oscillating with some frequency $\omega_\circ$ around the bottom of the well (*cf.* Fig. 1).

However, *if the "particle" is genuinely a quantum-mechanical system*, then, at least in principle, some quantum-mechanical effects might take a part in its dynamics, really breaking (or correcting) the deterministic behavior. And we are particularly interesting in the quantum-mechanical tunneling. This is a natural guess also in the programs (*cf.*, *e.g.*, Leggett [7]) of investigating the quantum-mechanical behavior of the macroscopic ("classical") physical systems: if the tunneling can ever occur, it proves unavoidable.

The issue whether the tunneling can ever occur in the associative neural networks is really a subtle question. Here we assume the possibility of the occurrence of the tunneling, and investigate the consequences. It proves that the tunneling in the associative neural nets can be dynamically modeled by the Pauli master equation [8], to be analytically solved in the next section.

2.1 Quantum-mechanical tunneling

The main question concerning the quantum-mechanical tunneling in macroscopic systems is whether the tunneling can ever occur in a system, whose behavior is usually considered to be fully deterministic ("classical"). This question is a matter of both, of principle, and of the details in the model of the system. It is almost a general strategy in investigating the quantum-mechanical effects in macroscopic systems to adopt, as a matter of principle, that the ultimate physical nature of the macroscopic systems is its the quantum-mechanical nature – the above mentioned universal validity of quantum mechanics. And then, it is a matter of the details in a model of the system to investigate the occurrence of some typical quantum effects. This is exactly our strategy distinguished in Section 1.

The standard (textbooks') physical situation concerning the tunneling assumes that the "particle" is in a quantum state, the mean energy of which is insufficient for overcoming the existing potential-energy barrier, and which is therefore classically bounded by the barrier. The tunneling, certainly, assumes the particle's "escape" to the continuum (free particle), which (contrary to the activation process [10]) is *not caused by the external stimuli.*

However, as it is clearly presented by Figure 1, the tunneling between the "patterns" represents a *tunneling between the bound states.* Being the tunneling between the macroscopically distinguishable states, it is analogous to the well-known MQT (macroscopic quantum tunneling) effect in the superconducting devices (SQUIDs, and CBJs) [7,9]. Unfortunately, while the MQT effect is followed by the highly semiclassical behavior [7], the tunneling between the "patterns" does not terminate with the semiclassical dynamics. To illustrate this, let us consider the $q - V$ plot in Figure 2.

Let us assume that, as presented in Figure 2b, the "particle" is initially in the well q_1. Its tunneling to, for instance, the well q_2, represents a stochastic "quantum jump", *i.e.* an indeterministic transition $q_1 \to q_2$. Once being in the well q_2, the *"particle" can proceed in tunneling, e.g.*, from q_2 to the well q_3, or (with some probability) to turn back to the well q_1, and so on. Certainly, this sequence of the tunnelings stops with the elapse of the time interval $\tau_{\rm stab}$.

The tunneling probability, $P_{ij}(t)$ (i-initial, j-final state), depends on the details in the model. First, so as to be able to see whether the tunneling is possible at all, one should [7] precisely know the values of the parameters (*cf.* Fig. 1), $V_\circ, \delta q, \omega_\circ$. Unfortunately, this is *not generally the case.* Second, it is also a matter of relative depth of the wells. Particularly: as it is well-known [10], if the initial well is "deeper" than the one to which the "particle" should tunnel to, the transition requires *external influence* (the, so-called, activation process), which is purely the

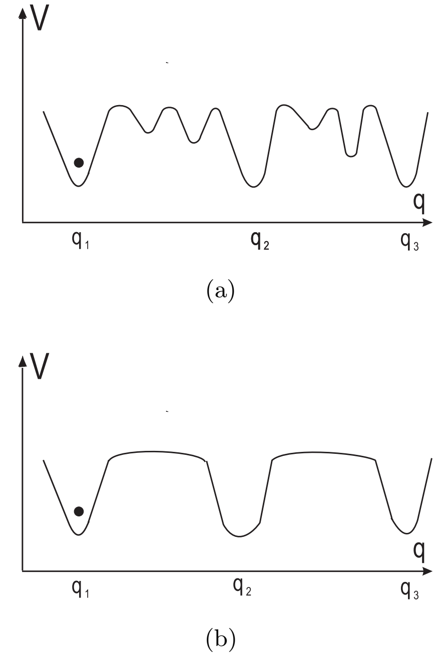

Fig. 2. A realistic $q - V$ plot, (a), is simplified (idealized) by (b). To justify this idealization it *suffices* that the minima in (a) are of only approximately equal depth, while $|q_{i\pm1} - q_i| \approx \delta q, \forall i$. Then one may forget about the particular shape of the plot between the minima (q_1, q_2, q_3), just taking care about the height of the potential barriers, likewise the mutual distances between the minima, in the given physical units. Certainly (*cf.* Sect. 4), if there are no such (mutually approximately equal-depth) minima, this model cannot be employed.

classic-physics effect: then the "particle" picks up some amount of energy from the "heat bath", necessary for the energy balance. Therefore, the tunneling can occur if the two wells are of approximately equal depth, or when the initial well is of the less depth than the "final" well – which is known as the "quantum hopping effect" [7].

Due to the fact that the necessary details concerning the form of the $q - V$ plot are not generally known, we shall restrict ourselves to the cases qualitatively presented by Figure 2b: a few, relatively close, equal-depth potential minima. Certainly, this is an idealization, but its relevance concerning the realistic cases will be distinguished in Section 4.

Finally, it is important to emphasize: the tunneling we are concerned with refers exactly to the transitions between the "patterns", and does not take into account the transitions from/to the coherent states, which represent the quantum-mechanical interferences of the macroscopically distinguishable states (which would be an analogue of the MQC (macroscopic quantum coherence) effect [11]). This is in accordance with the assumption (*cf.* above and in Introduction) that the neural networks represent the macroscopic physical systems. According to the theory of decoherence [12], the macroscopic systems are *open quantum systems* which are the objects of the decoherence effect [7,12], which destroys the superpositions (coherence) of the macroscopic states on a rather short timescale. (Note: the effect of decoherence *on the tunneling process* will be distinguished in Sect. 4.)

2.2 Relevance of the Pauli master equation

As it was distinguished above, in the time interval $\tau_{\rm stab}$, the tunneling (if it can occur at all) proves unavoidable. Actually, one may expect the tunneling effect to occur in a rather short time interval, $\tau_{\rm tun}$; more precisely, by $\tau_{\rm tun}$ we assume the time interval in which the tunneling probability, $P_{ij}(t)$, becomes comparable with unity.

If the ratio $\tau_{\rm stab}/\tau_{\rm tun}$ is neither greater, nor of the order of unity, there is a finite probability for the "particle" to make few the transitions (for instance, $q_1 \rightarrow q_2$, then $q_2 \rightarrow q_3$, etc.). On the other hand, if one may state $\tau_{\rm stab}/\tau_{\rm tun} \gg 1$, then the "particle" can make many (in the limit $\tau_{\rm stab}/\tau_{\rm tun} \rightarrow \infty$, infinitely many) the transitions.

The sequence of these transitions represents a *stochastic process*: the well-known "random walk" process. Further, we shall be concerned only with the finite number of the potential wells, which makes the methods of the "renormalization group" inappropriate, and really unnecessary.

Actually, the dynamics is such that: given an initial state q_i, with (nonzero) probability, P_{ij}, the "particle" can be found in the state q_j in an instant t, and then, in an instant $t' > t$, with (nonzero) probability, P_{jk}, to make transition to the state q_k, etc. Fortunately, this kind of the stochastic process can be *dynamically modeled* by the, so-called, Pauli master equation [8]:

$$\frac{{\rm d}P_m}{{\rm d}t} = \sum_{n(\neq m)} W_{nm}P_n - \sum_{n(\neq m)} W_{mn}P_m. \qquad (1)$$

In equation (1), P_m represents the probability of finding the "particle" in the state (potential well) m in an instant t, while W_{mn} represents the tunneling "velocity":

$$W_{mn} = \frac{{\rm d}P_{mn}}{{\rm d}t}, \qquad (2)$$

where P_{mn} represents a (time dependent) probability for tunneling from the state m to state n. It is usually considered:

$$W_{mn} = W_{nm}, \qquad (3)$$

the well-known microreversibility condition. Certainly, there is the constraint:

$$\sum_i P_i = 1, \forall t. \qquad (4)$$

Clearly, equation (1) distinguishes increase of P_m due to the transitions $q_n \rightarrow q_m, n \neq m$, and the decrease of P_m due to the transitions $q_m \rightarrow q_n, n \neq m$.

2.3 The task

The problem of investigating the quantum-mechanical tunneling in the associative neural networks reduces to the formal task of solving a set of the mutually coupled the Pauli master equations. Particularly, we are interesting in the next problem: *which form(s) of the $q - V$ plot provides a basis for minimizing the quantum fluctuations (due to the tunneling) in the information processing in the networks?*

As a quantitative measure of the quantum fluctuations it is convenient to use the Gibbs-von Neumann entropy, S [13]:

$$S = -k \sum_i P_i \ln P_i, \qquad (5)$$

where P_i represents the probability of finding the "particle" in the ith well in an instant t. Therefore, the task of "minimizing" the quantum fluctuations is really a task of "minimizing" the increase of entropy – which we shall be concerned with in Section 3.

3 Solutions of the Pauli master equations

Due to the lack of precise quantitative informations concerning the form of the $q - V$ plots (Figs. 1 and 2), we shall consider some typical $q - V$ models. For the sake of simplifying the calculation, we shall introduce some simplifications to be justified in Section 4.

3.1 Simplifications

We deal with the next plausible simplifications concerning the general form of the Pauli master equation: (i) we assume that only the transitions between the neighbor wells are effective (*i.e.*, we shall not consider the transitions, *e.g.*, $q_1 \rightarrow q_3$, etc.), (ii) we assume that $W_{mn} = W =$ const.,$\forall m, n$, and (iii) the ratio $\tau_{\rm stab}/\tau_{\rm tun} \sim 10$.

Then the task reduces to solving the set of the coupled linear differential equations of the type:

$$\begin{aligned}\frac{{\rm d}P_m}{{\rm d}t} &= W_{m+1,m}P_{m+1} + W_{m-1,m}P_{m-1} \\ &\quad - W_{m,m+1}P_m - W_{m,m-1}P_m \\ &= W(P_{m+1} + P_{m-1} - 2P_m). \qquad (6)\end{aligned}$$

Having the analytical expressions of the probabilities P_m, we shall numerically calculate their values in the instant $\tau_{\rm stab}$, thus obtaining the possibility for calculating the corresponding values of entropy, S, whose increase is due to the tunneling process. (Note: according to the above point (iii), in the interval $\tau_{\rm stab}$, on average, the system can make ten transitions.)

3.2 Solving the equation (6)

For the future purpose, and without loss of generality, we shall be concerned with the next cases: (a) three equal-distant, equal-depth wells, (b) six equal-distant equal-depth wells, and (c) three mutually "distant" groups of the pairs of equal-depth wells.

3.2.1 The case (a)

We consider the three minima (*cf.* Fig. 2b), q_1, q_2, q_3, $q_{i\pm 1} - q_i = \delta q$, $\forall i, j$. Then equation (6) leads to the next set of equations:

$$\begin{aligned}\dot{P}_1 &= W(P_2 - P_1)\\ \dot{P}_2 &= W(P_1 + P_3 - 2P_2)\\ \dot{P}_3 &= W(P_2 - P_3)\end{aligned} \tag{7}$$

while $P_1 + P_2 + P_3 = 1$, and $\dot{P}_i \equiv \mathrm{d}P_i/\mathrm{d}t$.

This set of equations can be easily solved. For instance, substituting $P_1 + P_3 = 1 - P_2$ in equation (7), the second equation becomes:

$$\dot{P}_2 = W - 3WP_2. \tag{8}$$

So one obtains:

$$P_2 = 1/3(1 - C_1 \exp(-3Wt)). \tag{9}$$

Further, one directly obtains from (7):

$$\frac{\mathrm{d}(P_1 - P_3)}{\mathrm{d}t} = -W(P_1 - P_3),$$

which leads to the solution:

$$P_1 - P_3 = B_1 \exp(-Wt). \tag{10}$$

From equations (9, 10) it directly follows:

$$\begin{aligned}P_1 &= 1/3 + (C_1/6)\exp(-3Wt) + (B_1/2)\exp(-Wt)\\ P_2 &= 1/3 - (C_1/3)\exp(-3Wt)\\ P_3 &= 1/3 + (C_1/6)\exp(-3Wt) - (B_1/2)\exp(-Wt).\end{aligned} \tag{11}$$

Let us consider the different the initial conditions: $P_1(t = 0) = 1$, and $P_2(t = 0) = 1$.

Respectively, one obtains:

$$\begin{aligned}P_1 &= 1/3 + (1/6)\exp(-3Wt) + (1/2)\exp(-Wt)\\ P_2 &= 1/3 - (1/3)\exp(-3Wt)\\ P_3 &= 1/3 + (1/6)\exp(-3Wt) - (1/2)\exp(-Wt)\end{aligned} \tag{12}$$

and

$$\begin{aligned}P_1 &= 1/3 - (1/3)\exp(-3Wt)\\ P_2 &= 1/3 + (2/3)\exp(-3Wt)\\ P_3 &= 1/3 - (1/3)\exp(-3Wt).\end{aligned} \tag{13}$$

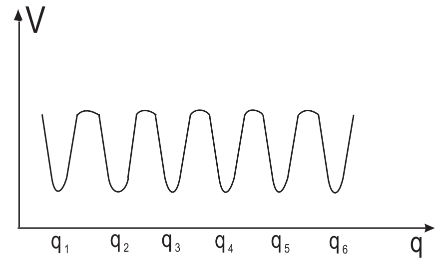

Fig. 3. Six, equal-depth, equal-distance minima; *i.e.*, $|q_{i\pm 1} - q_i| = \delta q, \forall i$.

It is obvious that in the limit $t \to \infty$ (which is equivalent with the limit $\tau_{\text{stab}}/\tau_{\text{tun}} \to \infty$) both sets of the solutions lead – as it can be plausibly expected – to the "equilibrium" probability distribution:

$$P_i = 1/3, \forall i. \tag{14}$$

This probability distribution corresponds to the maximal value of entropy:

$$S_{\max} = k \ln 3. \tag{15}$$

However, for $\tau_{\text{stab}}/\tau_{\text{tun}} = 10$, it should be numerically calculated. To this end it is convenient (and sufficient) to note that the product Wt can be estimated as:

$$Wt = \frac{1}{T}t. \tag{16}$$

This convenience follows from the next reasoning: if W can be considered as the average tunneling "velocity", then T represents interval for which the tunneling probability approximately equals unity; that is, $T = \tau_{\text{tun}}$. Now, setting $t = \tau_{\text{stab}}$, due to the point (iii) of Section 3.1, instead of the product Wt one should simply put the numerical value 10.

With this substitution one obtains the value:

$$S \approx 1.099k, \tag{17}$$

for both solutions, (12) and (13).

It is interesting to note that, up to the numerical error, one may state $\ln 3 \approx 1.099$. That is, in this case the system approaches the maximal value of entropy.

3.2.2 The case (b)

We are concerned with the six equal-distant, equal-depth wells, qualitatively presented by Figure 3.

Then equation (6) leads to the next set of the equations:

$$\begin{aligned}\dot{P}_1 &= W(P_2 - P_1)\\ \dot{P}_2 &= W(P_1 + P_3 - 2P_2)\\ \dot{P}_3 &= W(P_2 + P_4 - 2P_3)\\ \dot{P}_4 &= W(P_3 + P_5 - 2P_4)\\ \dot{P}_5 &= W(P_4 + P_6 - 2P_5)\\ \dot{P}_6 &= W(P_5 - P_6). \end{aligned} \quad (18)$$

This set of equations can be solved in few ways.

For instance, one may introduce the next variables:

$$x = P_1 + P_6, \quad y = P_2 + P_5, \quad z = P_3 + P_4, \quad (19a)$$
$$a = P_1 - P_6, \quad b = P_2 - P_5, \quad c = P_3 - P_4, \quad (19b)$$

with the inverse relations:

$$P_1 = (x + a)/2, P_6 = (x - a)/2, P_2 = (y + b)/2, \quad (20a)$$
$$P_5 = (y - b)/2, P_3 = (z + c)/2, P_3 = (z - c)/2. \quad (20b)$$

Then equations (18) transform into the next set of the differential equations:

$$\begin{aligned}\dot{x} &= W(y - x)\\ \dot{y} &= W - 3Wy\\ \dot{z} &= W(y - z)\end{aligned} \quad (21a)$$

while

$$\begin{aligned}\dot{a} &= W(b - a)\\ \dot{b} &= W(a + c - 2b)\\ \dot{c} &= W(b - 3c).\end{aligned} \quad (21b)$$

Equations (21a) can be solved in full analogy with the case (a), thus leading to the solutions:

$$\begin{aligned}x &= 1/3 + (\alpha_1/6)\exp(-3Wt) + (\alpha_2/2)\exp(-Wt)\\ y &= 1/3 - (\alpha_1/3)\exp(-Wt)\\ z &= 1/3 + (\alpha_1/6)\exp(-3Wt) - (\alpha_2/2)\exp(-Wt).\end{aligned} \quad (22)$$

Yet, the equations (21b) cannot be analogously solved. But bearing in mind the form of the above solutions, one may employ the Fourier transform, looking for the solutions in the form:

$$a(t) = \int a(q)\exp(-qt)\mathrm{d}q, \quad (23)$$

and analogously for $b(t)$ and for $c(t)$.

Substituting equation (23) into equation (21b), and after some algebra, one obtains the set of the linear algebraic homogeneous equations. The nontrivial solutions to these equations correspond to the next set of the values of q:

$$q \in \{0.268W; 2W; 3.732W\},$$

which give the solutions:

$$\begin{aligned}a(t) &= -\alpha\exp(-2Wt) - 0.366\beta\exp(-3.732WT)\\ &\quad + 0.366\gamma\exp(-0.268Wt)\\ b(t) &= \alpha\exp(-2Wt) + \beta\exp(-3.732Wt)\\ &\quad + \gamma\exp(-0.268Wt)\\ c(t) &= -\alpha\exp(-2Wt) + 1.366\beta\exp(-3.732Wt)\\ &\quad + 0.366\gamma\exp(-0.268Wt).\end{aligned} \quad (24)$$

Substitution of equations (22, 24) into equations (20) is straightforward. Let us therefore consider the two different the initial conditions, $P_1(t = 0) = 1$, and $P_3(t = 0) = 1$.

Respectively (and up to the numerical error), one obtains:

$$\begin{aligned}P_1 &= 1/6 + (1/12)\exp(-3Wt) + (1/4)\exp(-Wt)\\ &\quad + 0.211\exp(-2Wt) + 0.106\exp(-3.732Wt)\\ &\quad + 0.183\exp(-0.268Wt),\\ P_6 &= 1/6 + (1/12)\exp(-3Wt) + (1/4)\exp(-Wt)\\ &\quad - 0.211\exp(-2Wt) - 0.106\exp(-3.732Wt)\\ &\quad - 0.183\exp(-0.268Wt),\\ P_2 &= 1/6 - (1/6)\exp(-3Wt) - 0.211\exp(-2Wt)\\ &\quad - 0.289\exp(-3.732Wt) + 0.5\exp(-0.268Wt),\\ P_5 &= 1/6 - (1/6)\exp(-3Wt) + 0.211\exp(-2Wt)\\ &\quad + 0.289\exp(-3.732Wt) - 0.5\exp(-0.268Wt),\\ P_3 &= 1/6 + (1/12)\exp(-3Wt) - (1/4)\exp(-Wt)\\ &\quad + 0.211\exp(-2Wt) - 0.394\exp(-3.732Wt)\\ &\quad + 0.183\exp(-0.268Wt),\\ P_4 &= 1/6 + (1/12)\exp(-3Wt) - (1/4)\exp(-Wt)\\ &\quad - 0.211\exp(-2Wt) + 0.394\exp(-3.732Wt)\\ &\quad - 0.183\exp(-0.268Wt),\end{aligned} \quad (25)$$

and

$$\begin{aligned}P_1 &= 1/6 + (1/12)\exp(-3Wt) - (1/4)\exp(-Wt)\\ &\quad + 0.155\exp(-2Wt) - 0.106\exp(-3.732Wt)\\ &\quad - 0.049\exp(-0.268Wt),\\ P_6 &= 1/6 + (1/12)\exp(-3Wt) - (1/4)\exp(-Wt)\\ &\quad - 0.155\exp(-2Wt) + 0.106\exp(-3.732Wt)\\ &\quad + 0.049\exp(-0.268Wt),\\ P_2 &= 1/6 - (1/6)\exp(-3Wt) - 0.155\exp(-2Wt)\\ &\quad + 0.289\exp(-3.732Wt) - 0.134\exp(-0.268Wt),\\ P_5 &= 1/6 - (1/6)\exp(-3Wt) + 0.155\exp(-2Wt)\\ &\quad - 0.289\exp(-3.732Wt) + 0.139\exp(-0.268Wt),\\ P_3 &= 1/6 + (1/12)\exp(-3Wt) + (1/4)\exp(-Wt)\\ &\quad + 0.155\exp(-2Wt) + 0.395\exp(-3.732Wt)\\ &\quad - 0.049\exp(-0.268Wt),\\ P_4 &= 1/6 + (1/12)\exp(-3Wt) + (1/4)\exp(-Wt)\\ &\quad - 0.155\exp(-2Wt) - 0.395\exp(-3.732Wt)\\ &\quad + 0.049\exp(-0.268Wt).\end{aligned} \quad (26)$$

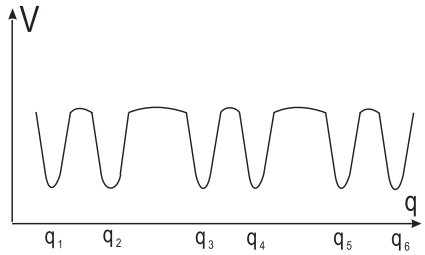

Fig. 4. Three groups of the pairs of equal-depth minima. Note: we assume that $q_1 - q_2 = q_3 - q_4 = q_5 - q_6$, while $q_2 - q_3 = q_4 - q_5 \gg q_1 - q_2$. This is also an idealization of the realistic cases, fully analogous to the idealization presented by Figure 2b.

It is obvious that in the limit $t \to \infty$, the both sets of the solutions lead to the "equilibrium" distribution:

$$P_i = 1/6, \forall i. \qquad (27)$$

which corresponds to the maximal value of entropy:

$$S_{\max} = k \ln 6 \approx 1.79k. \qquad (28)$$

However, for $Wt = 10$ one obtains, respectively,

$$S \approx 1.49k, \qquad (29)$$

and

$$S' \approx 1.79k. \qquad (30)$$

3.2.3 The case (c)

We consider the model qualitatively presented by Figure 4.

Note: we assume that the distances $q_2 - q_3$ and $q_4 - q_5$ are such that substantially lower the tunneling probability, and therefore the tunneling "velocity". That is, the tunneling in the pairs (the "zones"), (q_1, q_2), (q_3, q_4), (q_5, q_6), can be considered as substantially greater than the tunneling between the "zones" (*i.e.*, the transitions $q_2 \to q_3$, and $q_4 \to q_5$).

According to the simplifications in Section 3.1, we introduce $W_{12} = W_{34} = W_{56} = W = \text{const.}$, while $W_{23} = W_{45} = \omega \sim 0.01W$. Then equation (6) becomes:

$$\begin{aligned}
\dot{P}_1 &= W(P_2 - P_1)\\
\dot{P}_2 &= WP_1 + \omega P_3 - (W + \omega)P_2\\
\dot{P}_3 &= \omega P_2 + WP_4 - (W + \omega)P_3\\
\dot{P}_4 &= WP_3 + \omega P_5 - (W + \omega)P_4\\
\dot{P}_5 &= \omega P_4 + WP_6 - W(W + \omega)P_5\\
\dot{P}_6 &= W(P_5 - P_6). \qquad (31)
\end{aligned}$$

In full analogy with the case (b) one obtains the solutions:

$$\begin{aligned}
2P_1 &= C_1 + 1.015C_2 \exp(-0.015Wt)\\
&\quad - 0.995C_3 \exp(-2.005Wt) + 1.005C_4 \exp(-0.005Wt)\\
&\quad - C_5 \exp(-2Wt) - 0.985C_6 \exp(-2.015Wt)\\
2P_6 &= C_1 + 1.015C_2 \exp(-0.015Wt)\\
&\quad - 0.995C_3 \exp(-2.005Wt) - 1.005C_4 \exp(-0.005Wt)\\
&\quad + C_5 \exp(-2Wt) + 0.985C_6 \exp(-2.015Wt)\\
2P_2 &= C_1 + C_2 \exp(-0.015Wt) + C_3 \exp(-2.005Wt)\\
&\quad + C_4 \exp(-0.005Wt) + C_5 \exp(-2Wt)\\
&\quad + C_6 \exp(-2.015Wt)\\
2P_5 &= C_1 + C_2 \exp(-0.015Wt) + C_3 \exp(-2.005Wt)\\
&\quad - C_4 \exp(-0.005Wt) - C_5 \exp(-2Wt)\\
&\quad - C_6 \exp(-2.015Wt)\\
2P_3 &= C_1 - 2.015C_2 \exp(-0.015Wt)\\
&\quad - 0.005C_3 \exp(-2.005Wt) + 0.005C_4 \exp(-0.005Wt)\\
&\quad + C_5 \exp(-2Wt) - 2C_6 \exp(-2.015Wt)\\
2P_4 &= C_1 - 2.015C_2 \exp(-0.015Wt)\\
&\quad - 0.005C_3 \exp(-2.005Wt) - 0.005C_4 \exp(-0.005Wt)\\
&\quad - C_5 \exp(-2Wt) + 2C_6 \exp(-2.015Wt). \qquad (32)
\end{aligned}$$

For the initial condition $P_1(t = 0) = 1$ one obtains:

$$\begin{aligned}
&C_1 = 1/3, \quad C_2 \approx 1/6, \quad C_3 = -1/2,\\
&C_4 \approx 1/2, \quad C_5 \approx -0.334, \quad C_6 \approx -0.166,
\end{aligned}$$

while for $P_2(t = 0) = 1$:

$$\begin{aligned}
&C_1 = 1/3, \quad C_2 \approx 0.164, \quad C_3 \approx 1/2,\\
&C_4 \approx 1/2, \quad C_5 \approx 0.334, \quad C_6 \approx 0.168,
\end{aligned}$$

and for $P_3(t = 0) = 1$,

$$\begin{aligned}
&C_1 = 1/3, \quad C_2 \approx -0.33, \quad C_3 \approx -0.0025,\\
&C_4 \approx 0.0025, \quad C_5 \approx 0.332, \quad C_6 \approx -0.334.
\end{aligned}$$

These results correspond to the next values of entropy, respectively:

$$\begin{aligned}
S &= 0.844k\\
S' &= 0.898k\\
S'' &= 1.066k. \qquad (33)
\end{aligned}$$

Note: the maximal value of entropy is given by equation (28).

3.3 Analysis of the results

The cases (a)-(c) are typical, in the sense that they exhibit all the qualitative features of the process of the tunneling in the associative neural networks, involving the *finite number* of the "patterns" (potential wells). Actually, the number of the wells can be increased, but the main qualitative results maintain their respectability.

It is interesting to note that throughout the calculations, we need not to employ the concrete values of the tunneling "velocity", W. Rather, if the tunneling can occur at all, the occurrence of the process studied above is just *a matter of the time axes*, *i.e.*, of the ratio $\tau_{\mathrm{stab}}/\tau_{\mathrm{tun}}$. Nothing else is required – except the assumptions given in Section 3.1, to be discussed in Section 4.

Let us distinguish the physical role of entropy equation (5). As it is well-known [13], entropy represents a *measure of ignorance* about the physical state of the system. This interpretation directly follows [13] from equation (5). (Certainly, Eq. (5) follows from the von Neumann's entropy $S = -k\mathrm{tr}(\hat{\rho}\ln\hat{\rho})$ – $\hat{\rho}$ being the "density matrix" of the system (here: of the net) – where the "trace" operation is over the eigenstates of $\hat{\rho}$; here: the eigenstates are the quantum-mechanical counterpart of the "patterns", which represent the mutually distinguishable (*i.e., orthogonal*) states.) That is, entropy is a measure of the *classical indeterminism*: a system is in a definite state, but the state is not known with certainty, and *increase of entropy coincides with the loss of the informations concerning the physical state of the system*. As regards the associative neural nets, instead of being in a (classically stable, unique) state, *e.g.*, q_1, a net appears (in instant τ_{stab}) to be in another state – for instance, q_2. In what state the system can be found – *probabilistically* – is "measured" by entropy.

It is characteristic for all the stochastic processes that in the limit $t \to \infty$, the system reaches the equilibrium probability distribution. That is, for each the net, after sufficiently long time interval, the system would be equally probably found in either of the "wells".

However, and this is another specific feature of our considerations, the ratio $\tau_{\mathrm{stab}}/\tau_{\mathrm{tun}}$ is finite, and therefore we are concerned with the *transitional processes* [14], rather than with the equilibrium state. It is important to note: as regards the transitional processes, there are no the statements of the general relevance. Rather, one should carefully investigate the behavior of the system for the different, finite time intervals/instants, for each model separately. Let us therefore focus on the results obtained in the previous subsection.

The expressions (12, 13), (25, 26), (32), exhibit dependence of the probability distribution(s) upon the initial state of the system. It can be shown (which will be omitted here) that the probability distribution(s) exhibit somewhat unexpected behavior. (*E.g.*, one may obtain in the case (b) – *cf.* Sect. 3.2.2 – that some probability in the interval $[0, \tau_{\mathrm{stab}}]$ exhibit existence of the *maximal value(s)*, which hardly can be considered expectable from the only qualitative considerations – rather, one would expect that the probabilities monotonically approach to the equilibrium distribution.) As regards the increase of entropy, the results exhibit the expected characteristics for each case separately. *E.g.*, the expressions (33) exhibit the lower increase of entropy if the initial state is closer to the edges of the $q - V$ plot.

As regards the comparison of the cases studied, one directly observes that increase of the number of the minima implies increase of the entropy – as it can be plausibly expected. And this is of special interest in physical modeling of the associative neural nets, particularly since the big number of the minima and their high "density" are welcome for enhancing the memory capabilities of the nets. This points to a need to obtain, at least a qualitative, notion on the possible forms of the $q - V$ plots, which "minimize" the increase of entropy.

The case (c) clearly points to – in this sense – preferable form of the $q - V$ configuration(s) – the "zone" structure: *i.e.*, to mutually distant groups of not-very-numerous densely "packed" minima. In relatively short time intervals (*cf.* point (iii) of Sect. 3.1, and further Sect. 4), one may expect relatively slow increase, and relatively small the final value of entropy. This can be directly seen by comparing the values (33) with the values (29), and (30) – the later being referred to the case (b), in which there are no the "zones". Alternatively, this can be also seen by noting that the probabilities of the neighbor wells in the "zones" are of mutually equal order of magnitude, while the probability ratios concerning the different "zones" are not.

4 Discussion

Our strategy in investigating the quantum-mechanical effects in the associative neural networks relies upon the assumption (*cf.* Introduction) that the nets can be considered as the macroscopic ("classical") physical systems. Physical background of this assumption is that the nets represent the *open quantum systems*. That is, we adopt the generally employed hypothesis of universally valid quantum mechanics, in which context the macroscopic behavior follows as a consequence of the influence of the environment. To this end it is important to stress: there are the two different levels of the influence of the environment – which must not be mutually identified. First, it is the effect of *decoherence*, which destroys the quantum-mechanical coherence on a very short time scale. Second, it is the influence of the environment on the (approximately) deterministic behavior of the system – which has previously "survived" the decoherence; this is exactly the influence distinguished in Section 1 which, in contrast to the decoherentization, is not continuous in time (thus allowing for considering $\tau_{\mathrm{stab}} > 0$).

Throughout the paper we suppose the occurrence of the tunneling in the networks. However, this is really a subtle question.

As it was distinguished in Section 2.1, in order to be sure in this concern, one should answer a few questions. First, one should know the values of the parameters, $\omega_\circ$, $V_\circ$ and δq – *cf.* Figure 1. Second, one should know the ratio $\tau_{\mathrm{stab}}/\tau_{\mathrm{tun}}$; certainly, if $\tau_{\mathrm{stab}}/\tau_{\mathrm{tun}}$ is greater than, or of the order of unity, then the tunneling does not occur. Third, one should know about the relative depth of the local minima from/to which the tunneling might occur.

With this regard, nonexistence of the necessary data put the obstacles in investigating the tunneling in associative neural nets; by definition, the tunneling does not require any external stimuli (which is contrary to the thermal activation [10]). In order to making at least a qualitative results in this concern, we have considered the *idealized, typical models* (the cases (a)-(c)), based on the simplifications (i)-(iii) of Section 3.1. Particularly, we have considered the minima of equal depth, so requiring no energy exchange between the "particle" and its environment. Similarly, we have *plausibly adopted* equal distance between the minima – which directly justifies the simplification (ii), in which $W_{mn} = W, \forall m, n$. It is important to note that these idealizations do not make the considerations only weakly concerned with the realistic processes. Actually, the models considered can be easily extended so as to refer to the realistic $q - V$ plots (*cf.* Fig. 2a). In general, the tunneling can interfere with the different stochastic processes (*e.g.*, the activation process [10], which formally has the same effect as the one here studied), including the MQC effect [11], and apparently the net effect would be increase of the quantum fluctuations. Therefore, the results obtained in Section 3 give the *"lower bound" of the possible, realistic quantum fluctuations*, whose analogue has been experimentally verified [7,9] on very low temperatures. Yet, the assumption of existence of the equal-depth minima is somewhat restrictive. For it assumes equally learned contents, which is a feature of the "well trained" nets, rather than of the general case.

It is important to note that the interval τ_{stab}, in general, can take both extremal values, $\tau_{\text{stab}} \ll 1$, likewise $\tau_{\text{stab}} \gg 1$, depending on a model and on the actual physical situation. On the other side, the interval τ_{tun} is usually considered to be very short on a macroscopic time-scale. However, due to the "openness" of the neural nets, and in accordance with the experience with MQT effect [7,9], one may expect that the interval τ_{tun} is typically *substantially longer* than for an isolated quantum system. This gives a background for the assumptions (i), and (iii) – that the ratio $\tau_{\text{stab}}/\tau_{\text{tun}}$ is *finite* (rather than $\tau_{\text{stab}}/\tau_{\text{tun}} \gg 1$, as for the isolated systems).

As it was clearly distinguished in Section 2.2, the tunneling in the associative neural nets can be dynamically modeled by the Pauli master equation. A few words are in order to justifying the self-consistency of our considerations. Actually, it is well-known [8] that the Pauli master equation bears irreversibility, and therefore – by definition – *refers only to the open systems*. On the other side, this equation does *not* apply to the extremal time intervals: for very short time intervals, or to the limit $t \to \infty$. First, this is in accordance with the statement (*cf.*, *e.g.*, Grigolini [8]) that validity of the Pauli master equation assumes the previous occurrence of the process of decoherence; and this puts the limitation on the lower time-bound. As regards the upper time bound, we have already distinguished it in Introduction: it is the time interval τ_{stab}, which needs not to be macroscopically long.

Therefore, the physical picture we are here concerned with is completely consistent: The interaction of the "particle" with its environment has the two aspects. First, it produces the decoherence effect, thus providing the (approximately) "classical" (deterministic) behavior of the system. So the "patterns" appear as the well-defined macroscopic states. Second, the influence drives the "classical" behavior of the "particle"; the average time duration of the "pauses" between them is denoted by τ_{stab}. So, the effects considered here take a part (relatively) long after the decoherentization has actually occurred, and in between the two external stimuli – in full agreement with the foundations (*cf.*, *e.g.*, Grigolini [8]) of the Pauli master equation.

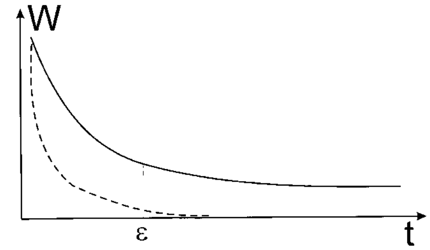

Fig. 5. The typical, *qualitative* dependence of the tunneling velocity, W, on time, t. Note that the interval ϵ is macroscopically very short. The solid line refers to the *open*, while the dotted line refers to the *isolated* quantum systems. For clarity, the dotted line is not put as it should – very close to the vertical axis.

Now one directly justifies the simplification (ii) of the Section 3.1: there we have assumed the time independence of the tunneling "velocity", W. Actually, as it is well-known [7], even for the open quantum systems the acceleration of the tunneling process is exponential in time, bearing the substantial change of the probability in the very short time intervals. Later, the rate of change of the tunneling probability, *in absolute units*, is very small, so one can approximate this change (at least in the zeroth order of approximation), and its "velocity", W, by a constant value. This is qualitatively presented by Figure 5. Note: the significant change of the tunneling "velocity" is during the time intervals of the order of a very short interval, ϵ. Then the absolute rate of change of W is rather small – see the almost flat part of the plot. And, as distinguished above, the Pauli master equation even does not apply for the intervals of the order of ϵ, but rather to the intervals corresponding to the almost flat part of the plot in Figure 5 (right from "ϵ"), for which W is of the almost constant value.

Finally, throughout the paper we have been concerned with the one-dimensional model of the "particle". However, as it was distinguished in Introduction, the physical state of a neural net is presented by a vector $\mathbf{q}$ – a "configuration" – which raises the task of the full formal analysis of the tunneling process. Whilst the calculation would complicate (a "pattern" is surrounded by more than just two neighbor "patterns"), the qualitative results obtained here maintain their respectability.

5 Conclusion

We investigate the quantum-mechanical tunneling between the "patterns" of the associative neural networks. This is really a macroscopic quantum effect which, if ever takes place, should be considered unavoidable. The net effect of the tunneling between the "patterns" is a stochastic process, the well-known "random walk" process, but rather on a finite number of the possible "positions" (here: the "patterns"). This dynamics can be modeled by the Pauli master equation, and is likely to be referred to the "transitional processes" – rather than to be concerned with the limit $t \to \infty$ (which here also would lead to the "equilibrium" probability distribution).

By analyzing the typical physical models, we obtain that the, so-called, "zone" structure of the "configuration-energy $(q - V)$" surfaces provides "minimizing" of the quantum fluctuations, which are due to the tunneling. By the "zone" structure we mean existence of the mutually "distant" groups of the dense, approximately equal-depth minima ("patterns"). This qualitative result is likely the most that can be told, so far. For more informations it is necessary to obtain some quantitative notions on the $q - V$ surfaces of the associative neural networks, which could also help in making theoretical predictions concerning distinguishing (*cf.* Ref. [7]) between the "pure" tunneling and the classical activation process.

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